Tiny primer on Gowers' solution to Banach's hyperplane problem

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April 8, 2012

Abstract

This is a small, simple extract from my thesis [1] that may serve as a minimal primer about Gowers' solution to Banach's hyperplane problem.

1 Preliminaries

Given a set B, we write $A \Subset B$ to indicate that A is a finite subset of B. We denote by $\mathbb{R}^{\mathbb{N}}$ the vector space of all sequences over $\mathbb{R}$;

$$c_{00} := \big\{(\alpha_1, \alpha_2, \ldots) \in \mathbb{K}^{\mathbb{N}} : \exists N \in \mathbb{N}\ \forall n \geqslant N : \alpha_n = 0\big\}$$

is the subspace of $\mathbb{R}^{\mathbb{N}}$ of finitely supported sequences. The vectors

$$e_n := (0, 0, \ldots, 0, \underset{\text{pos. } n}{1}, 0, 0, \ldots) \qquad (n \in \mathbb{N}) \tag{1}$$

form a Hamel basis for c_{00}, called the *standard unit vector basis.*

Remark 1.1. Let X and Y be normed spaces, and let $\Gamma_1 \colon X \to Y$ and $\Gamma_2 \colon Y \to X$ be bounded operators with $\Gamma_1\Gamma_2 = I_Y$. Then $P := \Gamma_2\Gamma_1$ is a bounded idempotent operator on X whose image is isomorphic to Y. More precisely, we have $\operatorname{im} P = \operatorname{im} \Gamma_2$, and the 'corestriction' of Γ_2 to $\operatorname{im} P$ (that is, the operator $y \mapsto \Gamma_2 y,\ Y \to \operatorname{im} P$) is an isomorphism; its inverse is the restriction of Γ_1 to $\operatorname{im} P$.

1.2. *Shift operators.* Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence. We associate with σ two operators on c_{00}, the *left shift* Λ_σ and the *right shift* R_σ, given by

$$\Lambda_\sigma x := \sum_{n=1}^{\infty} \alpha_{\sigma(n)} e_n \qquad \text{and} \qquad R_\sigma x := \sum_{n=1}^{\infty} \alpha_n e_{\sigma(n)} \qquad \Big(x = \sum_{n=1}^{\infty} \alpha_n e_n \in c_{00}\Big). \tag{2}$$

They are clearly linear, and $\Lambda_\sigma R_\sigma = I_{c_{00}}$, while $R_\sigma \Lambda_\sigma$ is the restriction to c_{00} of the projection $P_{\sigma(\mathbb{N})}$.

Definition 1.3. We say that the Schauder basis $(e_n)_{n\in\mathbb{N}}$ *admits uniformly bounded left shifts* if there exists some $K > 0$ such that $\|\Lambda_\sigma\| \leqslant K$ for all strictly increasing sequences σ.

2 Gowers' solution to the hyperplane problem

The Banach space G was introduced by Gowers as a counterexample to Banach's hyperplane problem [2] that asked: is every Banach space isomorphic to a subspace of its hyperplanes? 'Almost' in the Gowers–Maurey family [3], G has only operators of the form 'diagonal plus strictly singular'.

For the purposes of [1, Proposition 7.4.1], we note that the standard unit vectors form an unconditional basis for G. The key property of G that answers Banach's question in the negative is that G has no proper subspace isomorphic to itself.

The norm $\|\cdot\|_G$ of G is defined inductively. Define $G_N := \overline{(c_{00}, \|\cdot\|_{G_N})}$ with $\|x\|_{G_0} := \|x\|_\infty$, and, for $N \in \mathbb{N}_0$, $\|x\|_{G_{N+1}} := \max(\|x\|_{G_N^a}, \|x\|_{G_N^b})$ with

$$\|x\|_{G_N^a} := \sup\left\{ F(n)^{-1} \sum_{i=1}^{n} \|P_{E_i} x\|_{G_N} : n \in \mathbb{N}, E_1 < \cdots < E_n, E_i \Subset \mathbb{N} \right\}$$

(where $E_i < E_j$ is understood to mean $\max E_i < \min E_j$) and

$$\|x\|_{G_N^b} := \sup\left\{ |\langle x, g \rangle| : k \in K, g \in B_k^*(G_N) \right\},$$

with the following specially defined terms found in [2], whose definitions we omit: a function $F : \mathbb{R} \to \mathbb{R}^+$; a subset K of $\mathbb{N}$ and, for a normed space X and $k \in K$, a set of functionals $B_k^*(X) \subseteq X'$. The Banach space G is then defined to be the completion of c_{00} with respect to the norm $\|x\|_G := \lim_{N\to\infty} \|x\|_{G_N}$.

Example 2.1. The standard unit vector basis for the Banach space G, Gowers' solution to the hyperplane problem, admits uniformly bounded left shifts, but the right shift R_σ is not norm-bounded for any non-trivial σ.

Sketch proof. Let τ be an increasing sequence. The norm is defined inductively, hence we prove by induction that $\|\Lambda_\tau\| \leqslant 1$ on each G_N for $N \in \mathbb{N}_0$. It is clearly true for G_0. Now assume it has been proven for N, and let subsets $E_1 < \cdots < E_n$ of $\mathbb{N}$ be given: then $\tau(E_1) < \cdots < \tau(E_N)$ and $\|P_{E_i} \Lambda_\tau x\|_{G_N} = \left\|P_{\tau(E_i)} x\right\|_{G_N}$. Hence $\|\Lambda_\tau x\|_{G_N^a} \leqslant \|x\|_{G_N^a}$.

We claim that $B_k^*(\Lambda_\tau G_N) \subseteq B_k^*(G_N)$. Since $\|\Lambda_\tau x\|_{G_N} \leqslant \|x\|_{G_N}$ by the inductive hypothesis, the proof of this claim is similar to that of the following observation in [2]: that for normed spaces Y and Z, if $\|x\|_Y \leqslant \|x\|_Z$ for all $x \in c_{00}$, then $B_k^*(Y) \subseteq B_k^*(Z)$. Assuming the claim, it then follows that $\|\Lambda_\tau x\|_{G_N^b} \leqslant \|x\|_{G_N^b}$, and hence $\|\Lambda_\tau x\|_{G_N} \leqslant \|x\|_{G_N}$.

The second statement follows from the first by the main property of G: if R_σ were bounded for some (non-trivial) strictly increasing sequence σ ($\sigma \neq \iota$), then $\Lambda_\sigma R_\sigma = I$ and so by Remark 1.1, $R_\sigma G$ would be a proper subspace isomorphic to the whole of G, which gives a contradiction. □

References

[1] A. Bird, *A study of the James–Schreier spaces as Banach spaces and Banach algebras*, Ph.D. Thesis, Lancaster University, 2010.

[2] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. 26 (1994), 523–530.

[3] W. T. Gowers and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.